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9. Notes on a Method for Describing Definitions Thorems, Proofs in Mathematics

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1 Introduction

Future goal of our study is to build knowledge base concerning mathematics on a computer and to build a computer with which we can converse for mathematics. As first step for it, in this paper, we shall propose the notation for describing definitions, theorems, proofs and so on in mathematics which may be convenient to be stored in the computer as information base. Definitions of mathematical objects, theorems which express the relations of mathematical objects and their proofs should be represented by logical symbols originally. For example

(1) $\mathrm{Set}(X) \wedge \mathrm{Set}(Y) \wedge \forall x (x \in X \rightarrow x \in Y)$

(2) $\forall u (u \in Z \leftrightarrow (u \in X \vee u \in Y))$

$$(3)\ \forall X \forall Y \forall Z(X(\mathrm{set}) \wedge Y(\mathrm{set}) \wedge Z(\mathrm{set}) \wedge \forall u(u \in X \rightarrow u \in Y) \wedge \forall v(v \in Y \rightarrow v \in Z) \rightarrow \forall w(w \in X \rightarrow w \in Z))$$

where 'Set(x)' is an abbreviation form of the logical expression which means that X is a set. By using a usual mathematical language, we can express (1), (2) and (3) as follows:

(1)' A set X is a subset of a set Y,

(2)' Z is a union of X and Y,

and

(3)' If a set X is a subset of a set Y and a set Y is a subset of a set Z, then X is a subset of Y.

Essentially all mathematical notations should be expressed by using logical symbols (1), (2) and (3). And if we store mathematical data on a computer with this form, it seems to be convenient to investigate and refer these data. But comparing with usual mathematical representation as (1)'-(3)', the logical expression seems to be hard for writing and reading. So, in this paper, we shall propose the new notations concerned with mathematics between the logical expressions and usual mathematical expression. That is, it is not hard for research workers to read and write the mathematical expressions by our methods. And the expressions in our methods can be easily translate into logical symbols by a computer. Finally we shall examine about the ability of our method and about its processing on a computer.

2 <u>Method for describing the notations about mathematics</u>

We shall successively list the rules of method for describing the notations about mathematics---definitions, theorems, proofs and so on---.

2.1 Begin and End of the notation

When we describe the notation concerned with mathematics, first we should desctibe

Notation

and finally we should write

End

Therefore the notation should be always formed as follows:

Notation

p1

p2

p3

.

.

pn

End

where p1, p2,..., pn are called sentences which shall be explained from this. Specially when we want to name the notation, we should write as the following:

Notation of <name>

2.2 Declaration of free variables

When we want to use a certain symbol (for instance x) for a free variable, we should write

x:

which means that a symbol x is a free variable. This declaration is effective until the end of notation without an exception mentioned in a section 2.5. If we want to use many free variable (for instance x_1, x_2, and x_3), according to the above rule we write

$$\boxed{\begin{array}{l} x_1: \\ x_2: \\ x_3: \end{array}}$$

Moreover the following abbreviation may be used

$$\boxed{x_1,\ x_2,\ x_3:}$$

or

$$\boxed{x_i\ (i=1,\ 2,\ 3):}$$

More generally, the description as follows may be used

$$\boxed{x_i\,(i\in I):}$$

However, in this case, we must say before this what a set I is by using condition sentences mentioned the next section.

2.3 Describing result sentences and condition sentences

When every free variable in a logical expression P is declared before this by the method mentioned in 2.2 (therefore if P is a closed logical expression, we need not it), the followings are respectively called a result sentence (a) and condition sentence (b);

(a) $\boxed{\Rightarrow P}$ or $\boxed{P}$

(b) $\boxed{:P}$

EXAMPLE 1:

Let P(x) be a logical expression with free variable x. Then the following notation (1) and (2) respectively means that

(1) for any x, we have P(x)

(2) let x be a free variable satisfying P(x)

(1)
$$\boxed{\begin{array}{l} \quad\text{Notation} \\ x: \\ \qquad P(x) \\ \quad\text{End} \end{array}}$$

(2)
$$\boxed{\begin{array}{l} \quad\text{Notation} \\ x: \\ \qquad :P(x) \\ \quad\text{End} \end{array}}$$

EXAMPLE 2:

Let P(x) and Q(x) be logical expressions with free variable x. Then the following notation means that 'if x satisfies condition P(x), then we have Q(x).'

```
Notation
  x:
   :P(x)
    Q(x)
  End
```

Seeing in the above example, if there exist several condition sentences ':P1(x)', 'P2(x)', ..., 'Pn(x)' between the declaration sentence 'x:' and the condition sentence 'Q(x)', then we read it as follows:

$\forall x(P1(x) \wedge P2(x) \wedge \ldots \wedge Pn(x) \rightarrow Q(x))$

NOTE:

We admit the rule of the abbreviation as follows:

```
x:
  :P(x)        ⇒        x:P(x)
```

2.4 Naming the logical expression

We can give a name N to a logical expression P as follows:

$N \underset{<}{\equiv} P$

EXAMPLE 3

```
Notation
X: Set(x)
Y: Set(y)
 X⊂Y ≡ ∀x(x∈X→x∈Y)
 End
```

This natation means that 'Let X and Y be a set. When x belongs to Y for any x in X, we write X⊂Y.'

NOTE:

This naming is effective until the end of notation without an exception mentioned in a section 2.6.

2.5 Method for using the 'show that' sentence

When we would like to show the proof of the proposition P (that is, description of process for deriving P), we write as follows:

```
Show that P
    P1
    P2
    .
    .
    .
    .
    Pn
 Hence P
```

where P1, P2, ..., Pn are sentences which show a process for condition sentences in 'show that' sentence are effective until 'Hence'.

2.6 Descriptions of definitions theorems and proofs

The declaration of a definition must be written between 'Definition' and 'End' as follows:

```
Definition
    .
    .
    .
   End
```

Similarly the description of a theorem and a proof must be written as follows:

```
Theorem
   .
   .
   .
   .
   .
End
```

```
Theorem
   .
   .
 End
 Proof
   .
   .
 QED
```

REMARK:

If we want to name the definition, we may write as follows:

Definition of < name >

For the theorem and the proof, we write in the same way. If the theorem is written before the proof, we should write without fail as follows:

Proof of <theorem's name>

2.7 <u>The method of using 'from'</u>

We may write 'from P1, P2, ...Pn' immediately after a result sentence, where P1, p2,...Pn is respectively one of

(1) a sentence or a symbol which is mentioned before this result sentence in the same notation,

(2) a name of the definition or theorem.

This shows by what condition, theorem and definition the result sentence can be lead.

Lastly we shall show the example of notation which mean the proof of the proposition of (3) in chapter 1.

```
Notation
X: Set(X)
Y: Set(Y)
Z: Set(Z)
X⊂Y ⇌ ∀x(x∈X→x∈Y)
Y⊂Z ⇌ ∀y(y∈Y→y∈Z)
X⊂Z ⇌ ∀x(x∈X→x∈Z)
 :X⊂Y, Y⊂Z
Show that X⊂Z
 x: x∈X
 x∈Y   from x:x∈X, :X⊂Y
 x∈Z   from x:x∈Y, :Y⊂Z
 ∀x(x∈X→x∈Z) from x:x∈X, x∈Z
 Hence X⊂Z
 End
```

3. Description on definitions theorems and proofs

We shall show typical examples of descriptions on definitions, theorems and proofs.

3.1 Example of the description of definition

We can devide so called 'definition' into the following two classes:

(1) putting a sign on a certain mathematical object (for example, $\{x|\ a<x<b\}$ is denoted by $[a, b]$)

(2) naming a property of a mathematical object (for example, if n can be devided by 2, then n is said to be 'even')

CASE (1)

A mathematical object is generally defined by the following symbol expression

$$\forall x_1 \ldots \forall x_n (P(x_1, \ldots, x_n) \rightarrow \exists ! y (Q(x_1, \ldots x_n, y))$$

where symbol '!' means a uniqueness. We want to put a symbol (or name) $N(x_1, \ldots, x_n)$ to this mathematical object y. The above argument can be expressed by an ordinary mathematical language as follows:

'Let x_1, x_2, ..., x_n be mathematical objects satisfying the condition $P(x_1, \ldots, x_n)$. And there exists a unique mathematical object y such that $Q(x_1, \ldots, x_n, y)$. Then we put y a symbol $n(x_1, \ldots, x_n)$'

Therefore this notation is as follows:

Notation
Definition
$x_1, x_2, \ldots, x_n$: $P(x_1, x_2, \ldots, x_n)$
$\exists ! y (Q(x_1, x_2, \ldots, x_n, y))$
$y : Q(x_1, x_2, \ldots, x_n, y)$
$N(x_1, x_2, \ldots, x_n) \equiv y$
End
End

REMARK:

(1) the description

$\exists x (Q(x))$
$x : Q(x)$

can be abbreviated as follows:

$\boxed{\exists x : Q(x)}$

(2) we may use a Halmos symbol '□' for 'End'.

We shall define 'union' by our method as follows:

EXAMPLE 4:

Notation

Definition of union

Set(X)

Set(Y)

$\exists! U : U = \{x \mid x \in X \vee x \in Y\}$

$X \cup Y \underset{\leftarrow}{\equiv} U$ □

End

CASE (2)

Under the assumption that mathematical objects $x_1, \ldots x_n$ satisfy the condition $P(x_1, \ldots, x_n)$, we want to put the property $Q(x_1, \ldots, x_n)$ a symbol $n(x_1, \ldots, x_n)$. So we describe this as follows:

Notation

Definition

$N(x_1, \ldots, x_n) \underset{\leftarrow}{\equiv} Q(x_1, \ldots, x_n)$ □

End

EXAMPLE 5:

We show the notation of inclusion

Notation

Definition of inclusion

Set(X)

Set(Y)

$X \subset Y \underset{\leftarrow}{\equiv} \forall x (x \in X \rightarrow x \in Y)$ □

End

3.2 <u>Example of the description of theorem</u>

A theorem has generally the following form:

$x_1, \ldots, x_n : (P(x_1, \ldots, x_n) \rightarrow Q(x_1, \ldots, x_n))$

Therefore we describe a theorem as follows:

Notation

Theorem

$x_1, x_2, \ldots, x_n : P(x_1, \ldots, x_n)$

$\Rightarrow Q(x_1, \ldots, x_n)$ $\square$

End

EXAMPLE 6:

According to the above form, we describe the notation (3) in 1 as follows:

Notation

Theorem

$X : Set(X)$

$Y : Set(Y), \quad X \subset Y$

$Z : Set(Z), \quad Y \subset Z$

$\Rightarrow X \subset Z$ $\square$

End

Therefore the theorem which has the following form

$\forall x_1, \ldots, \forall x_n (P(x_1, \ldots, x_n) \rightarrow \forall y_1, \ldots, \forall y_m (Q1(x_1, \ldots, x_n, y_1, \ldots, y_m) \leftrightarrow$
$Q1(x_1, \ldots, x_n, y_1, \ldots, y_m) \leftrightarrow \ldots \; Ql(x_1, \ldots, x_n, y_1, \ldots, y_m))$

can be expressed as follows:

Notation

Theorem

$x_1, \ldots, x_n : P(x_1, \ldots, x_n)$

$y_1, \ldots, y_m :$

$\Rightarrow$

$Q1(x_1, \ldots, x_n, y_1, \ldots, y_m)$

$\leftrightarrow Q2(x_1, \ldots, x_n, y_1, \ldots, y_m)$

$\leftrightarrow Q3(x_1, \ldots, x_n, y_1, \ldots, y_m)$

.

.

.

$\leftrightarrow Q1(x_1,\ldots,x_n,y_1,\ldots,y_m)$ □

End

3.3 Example of description of Proof

A proof of a theorem fundamentally seems to be a description of process which show how the result of the theorem can be lead from the assumption of the theorem. Hence if we want to describe the proof immediately after the theorem, we must first write

Proof

and write the description of the proof. And if we want to describe the proof away from the theorem, we must write

Proof of <Name of Theorem>

And we should write in the end of the proof.

QED

According to the above argument, the notation of the proof has the following form:

```
Notation
Theorem
    .
    .
End
Proof
    .
    .
QED
End
```

Also, when a proof is separated from the theorem, we write the notation of the proof as follows:

Notation

Proof of <Name of theorem>

.

.

.

QED

End

In this case, assuming that the theorem which should be is put 'Notation' and 'Proof', we can use the free variables in the theorem.

EXAMPLE7:

We shall write the proposition (3) in chapter 1 and its proof.

Notation

Theorem

X,Y,Z:set

   :$X\subset Y$, $Y\subset Z$

$\Rightarrow$ $X\subset Z$

Proof

Show that $\forall x(x\in X\rightarrow x\in Z)$

x:$x\in X$

$x\in Y$ from x:$x\in X$, $X\subset Y$

$x\in Z$ from $x\in Y$, $Y\subset Z$

Hence $\forall x(x\in X\rightarrow x\in z)$

$X\subset Z$

QED

End

The notation described formally is sometimes lengthily and rather illegible. In this case it is convenient to use the abbriviated notation. Hence we shall list the table of the forms of the abbreviated notations in Appendix I. But it is possible to define the other forms of the abbreviated notation in addition to this in the case of need. Table of abbreviation is given in Appendix I.

In Appendix II, we shall show the example of the description which is written by the method that we propose in this paper. The theorem is fundamental in topology.

4 Discussions

We shall discuss our notations describing definitions, theorems, proofs and so on in mathematics. Especially, we shall consider the describability of this method and the possibility of constructing mathematical database based on these notations.

4.1 Describability and possibility

Firstly, we inspect the describability. For the purpose of supporting our proposal our method should have following two properties; many fields of mathematics can be represented in our notations and mathematicians easily read and write mathematical works in these notations.

We have written a lot of mathematical theorems and their proofs in our notations experimentally. Some example have been shown in this paper. Contents of undergraduate textbooks on set theory and topology have been easily written in our notations. We are now describing other fields on mathematics. Our experience shows that this method will be applicable to many fields of mathematics by adding a few notations. Moreover, our notations are similar to ordinary ones which are written on the blackboard or textbooks. Thus, it is expected that these notations are accepted by many mathematicians.

Secondly, we investigate the possibility of constructing mathematical data base on computer. Let us assume that we construct knowledge base of mathematics on a computer by using our notations. There are many varieties of ways to construct and use such database. That is, one way is to construct and use database as a book, another way is to construct knowledge base which can support a mathematician who discovers a new theorem and proves it. For example, if the mathematician can easily see whether the theorem is essentially same as the known theorem or not by using the knowledge base.

4.2 Consideration on the implementation

Moreover, if we build a computer system which can detect some errors in his proof, a system based on our notations will be a powerful tool for mathematicians. Building such a system is a future goal of our study, however, taking into considerations of existing computer technologies, we discuss several aspects of the implementation of knowledge base concerning mathematics in this paper. Considering hardware technologies, we can easily make a special computer or a terminal with character sets of our notations. But, in software technologies, there are many problems to construct a computer system which provides highly useful facilities for mathematicians. So we classify those facilities and discuss implementation methods of the knowledge base and algorithms retrieving a particular fact from the base.

The simplest case is that we directly store facts written in our notations and read them like a book. In this case, we retrieve a particular fact by using its name. Though we can easily implement such a system, this is just the same as a dictionary. However, our method is independent of a specific language. Thus, we can output the content of database in some languages, i.e. English, Japanese, French or German. If we construct a program translating into the specified language, the content of database if displayed in that lnaguage.

Next, let us consider that the database includes pattern variables and we retrieve or display the content by using pattern matching algorithms. For example, X and Y in Example 3 should match arbitrary names of sets, say A and B. These facilities provide the method for defining '0' in the example. When these facilities are implemented, the computer is useful for us. In the case, the database is not larger than the former one, but the algorithm retrieving facts becomes complicated and more memory space is needed. If we can construct such that a system including mathematical formulas, this will be a convenient tool for engineers and applied mathematicians.

Finally, we discuss the system which is useful for active researchers in mathematics. In this case, our system should provide higher facilities. For example, this system should provide an inference facility following the proof which is newly shown by a mathematician. Moreover, this system should give useful guidances to the mathematician, if he wants to know whether a theorem which is discovered by him is essentially equal to the known theorem or not. Since existing methodologies on automatic theorem proving and formula manipulation are insufficient to be used in constructing such a system, we have to develop new methods for our purpose.

5 Conclusions

We can implement the simplest system described above on a minicomputer or microcomputer by modifying character sets of a particular computer. We will be able to construct the second system on a personal computer in the near future. In order to construct the system with high facilities, it is necessary that many problems in computer science, especially, in software technology, are solved. These problems are attractive research objects for computer scientists.

Now we conclude that a pilot system based on our proposal should be implemented as soon as possible, and experiences with the implementation will give us useful guidances of improving our method.

Appendix I

It is convenient to use the abbreviation form, if the normal notation seems to be lengthy. In this appendix, we shall list up the rules for abbreviating the normal notations. In addition to these abbreviation forms, we can abbreviate the normal notaions in case of needs.

List of abbreiviation forms

normal notation	abbreviation form	remark
$\Rightarrow$ P	P	
x : : P(x)	x : P(x)	
. . . . End	. . . . $\square$	
$\exists$y P(y) y : P(y)	$\exists$y : P(y)	
x ; x = y $x{:}x(N(x_1 \ldots x_n))$	$x \equiv y$ $x{:}N(x_1 \ldots x_n)$	 $x{:}(N(x_1 \ldots x_n))$ means the abbreviation from which represents the property of x.

<table>
<tr><td>(example)

x: Set(x)</td><td>x: set</td></tr>
<tr><td>y_1: $y_1(N(x_1 \ldots x_n))$
y_2: $y_2(N(x_1 \ldots x_n))$
⋮
y_m: $y_m(N(x_1 \ldots x_n))$</td><td>$y_1 y_2 \ldots y_m$: $N(x_1 \ldots x_n)$</td></tr>
<tr><td>(example)
y_1: set(y_1)
y_2: set(y_2)</td><td>y_1, y_2: set</td></tr>
<tr><td>Show that $\forall x(P(x) \to Q(x))$
x: P(x)
⋮
Q(x)
Hence $\forall x(P(x) \to Q(x))$</td><td>If x: P(x)
⋮
then Q(x)</td></tr>
<tr><td>Show that $P \to Q$
:P
⋮
Q
Hence $P \to Q$</td><td>If : P
⋮
then Q</td></tr>
</table>

Appendix II

We shall represent the following fundamental theorem in topology and its proof by the method of notation which proposed in this paper.

THEOREM Let (X,T) be a topolofy space and A be a subset of X. Then A is an element of T, if and only if for any x A there exixts 0 T such that x 0 A.

Notation

Theorem

(X, T): topological space

A : $A \subset X$

$\Rightarrow$

$A \in T$

$\Leftrightarrow$

$(\forall x \in A)\ (\exists O \in T)[x \in O \subset A]$ $\Box$

Proof

Show that $A \in T \rightarrow (\forall x \in A)\ (\exists O \in T)\ [x \in O \subset A]$

x: $x \in A$

$A \in T$, $x \in A \subset A$

$\exists O$: $O = A$

$O \in T$, $x \in O \subset A$

$(\forall x \in A)\ (\exists O \in T)[x \in O \subset A]$

Hence $A \in T \rightarrow (\forall x \in A)\ (\exists O \in T)\ [x \in O \subset A]$

If $(\forall x \in A)\ (\exists O \in T)[x \in O \subset A]$

$\exists M$: map (A,T), $(\forall x \in A)\ [x \in M(x) \subset A]$

Show that $A = \bigcup_{x \in A} M(x)$

If $x : x \in A$, $x \in M(x)$

then $x \in \bigcup_{x \in A} M(x)$

$A \subset \bigcup_{x \in A} M(x)$

$(\forall x \in A)[M(x) \subset A]$

$\bigcup_{x \in A} M(x) \subset A$

Hence $A = \bigcup_{x \in A} M(x)$

then $A \in T$

QED

END